EXERCISES

1. Find the arc length of the following parametrized curves:
  a. $\mathbf{g}(t) = (a\cos t,\ a\sin t,\ bt)$, $t \in [0, 2\pi]$.
  b. $\mathbf{g}(t) = (\frac{1}{3}t^3 - t,\ t^2)$, $t \in [0, 2]$.
  c. $\mathbf{g}(t) = (\log t,\ 2t,\ t^2)$, $t \in [1, e]$.
  d. $\mathbf{g}(t) = (6t,\ 4t^{3/2},\ -4t^{3/2},\ 3t^2)$, $t \in [0, 2]$.

2. Express the arc length of the following curves in terms of the integral

$$E(k) = \int_0^{\pi/2} \sqrt{1 - k^2 \sin^2 t}\, dt \qquad (0 < k < 1),$$

for suitable values of k. ($E(k)$ is one of the standard **elliptic integrals**, so called because of their connection with the arc length of an ellipse.)
  a. An ellipse with semimajor axis a and semiminor axis b.
  b. The portion of the intersection of the sphere $x^2 + y^2 + z^2 = 4$ and the cylinder $x^2 + y^2 - 2y = 0$ lying in the first octant.

3. Find the centroid of the curve $y = \cosh x$, $-1 \le x \le 1$.

4. Compute $\int_C \sqrt{z}\, ds$ where C is parametrized by $\mathbf{g}(t) = (2\cos t,\ 2\sin t,\ t^2)$, $0 \le t \le 2\pi$.

5. Compute $\int_C \mathbf{F} \cdot d\mathbf{x}$ for the following $\mathbf{F}$ and C:
  a. $\mathbf{F}(x, y, z) = (yz,\ x^2,\ xz)$; C is the line segment from $(0,0,0)$ to $(1,1,1)$.
  b. $\mathbf{F}$ is as in (a); C is the portion of the curve $y = x^2$, $z = x^3$ from $(0,0,0)$ to $(1,1,1)$.
  c. $\mathbf{F}(x, y) = (x - y,\ x + y)$; C is the circle $x^2 + y^2 = 1$, oriented clockwise.
  d. $\mathbf{F}(x, y) = (x^2 y,\ x^3 y^2)$; C is the closed curve formed by portions of the line $y = 4$ and the parabola $y = x^2$, oriented counterclockwise.

6. Compute the following line integrals:
  a. $\int_C (xe^{-y}\, dx + \sin \pi x\, dy)$, where C is the portion of the parabola $y = x^2$ from $(0,0)$ to $(1,1)$.
  b. $\int_C (y\, dx + z\, dy + xy\, dz)$, where C is given by $x = \cos t$, $y = \sin t$, $z = t$ with $0 \le t \le 2\pi$.
  c. $\int_C (y^2 dx - 2x\, dy)$, where C is the triangle with vertices $(0,0)$, $(1,0)$, and $(1,1)$, oriented counterclockwise.

7. Let $\mathbf{F} : \mathbb{R}^n \to \mathbb{R}^m$ be a continuous map, and let C be a C^1 curve in $\mathbb{R}^n$.
  a. Deduce from Proposition 5.8 that $|\int_C \mathbf{F}\, ds| \le \int_C |\mathbf{F}|\, ds$.
  b. In the case $m = n$, show that $|\int_C \mathbf{F} \cdot d\mathbf{x}| \le \int_C |\mathbf{F}|\, ds$.

8. Prove in detail that arc length, as defined for rectifiable curves, is additive; that is, if C, C_1, and C_2 are the curves parametrized by $\mathbf{g}(t)$ for $t \in [a, b]$, $t \in [a, c]$, and $t \in [c, b]$, then $L(C) = L(C_1) + L(C_2)$.

9. Let $\mathbf{g}(t) = (g(t), h(t))$ be a C^1 parametrization of a plane curve. Given a partition $P = \{t_0, \ldots, t_J\}$ of $[a, b]$, the distance between two neighboring points $\mathbf{g}(t_{j-1})$ and $\mathbf{g}(t_j)$ is

$$\sqrt{[g(t_j) - g(t_{j-1})]^2 + [h(t_j) - h(t_{j-1})]^2}.$$

Use the mean value theorem to express the differences inside the square root in terms of g' and h', and then use Exercise 9 in §4.1 to give an alternate proof of Theorem 5.11. (Exactly the same idea works for curves in $\mathbb{R}^n$.)

5.2 Green's Theorem

Green's theorem is the simplest of a group of theorems — actually, they're all special cases of one big theorem, as we shall indicate in §5.9 — that say that "the integral of something over the boundary of a region equals the integral of something else over the region itself." To state it, we need some terminology.

A **simple closed curve** in $\mathbb{R}^n$ is a curve whose starting and ending points coincide, but that does not intersect itself otherwise. More precisely, a simple closed curve is one that can be parametrized by a continuous map $\mathbf{x} = \mathbf{g}(t)$, $a \le t \le b$, such that $\mathbf{g}(a) = \mathbf{g}(b)$ but $\mathbf{g}(s) \ne \mathbf{g}(t)$ unless $\{s, t\} = \{a, b\}$.

We shall use the term **regular region** to mean a compact set in $\mathbb{R}^n$ that is the closure of its interior. Equivalently, a compact set $S \subset \mathbb{R}^n$ is a regular region if every neighborhood of every point on the boundary ∂S contains points in S^{int}. For example, a closed ball is a regular region, but a closed line segment in $\mathbb{R}^n$ $(n > 1)$ is not, because its interior is empty.

Now let $n = 2$. We say that a regular region $S \subset \mathbb{R}^2$ has a **piecewise smooth boundary** if the boundary ∂S consists of a finite union of disjoint, piecewise smooth simple closed curves, where "piecewise smooth" has the meaning assigned in the previous section. (We thus allow the possibility that S contains "holes," so that its boundary may be disconnected.) In this case, the **positive orientation** on ∂S is the orientation on each of the closed curves that make up the boundary such that the region S is on the *left* with respect to the positive direction on the curve. More precisely, if $\mathbf{x}$ is a point on ∂S at which ∂S is smooth, and $\mathbf{t} = (t_1, t_2)$ is the unit tangent vector in the positive direction at that point, then the vector $\mathbf{n} = (t_2, -t_1)$, obtained by rotating $\mathbf{t}$ by 90° *clockwise,* points *out* of S. (That is, $\mathbf{x} + \epsilon\mathbf{n} \notin S$ for small $\epsilon > 0$.) See Figure 5.4.

If $\mathbf{F} = (F_1, F_2)$ is a continuous vector field on $\mathbb{R}^2$, we denote by

$$\int_{\partial S} \mathbf{F} \cdot d\mathbf{x} \quad \text{or} \quad \int_{\partial S} F_1\, dx_1 + F_2\, dx_2$$

Hence, by applying Green's theorem to the rotated field $\widetilde{\mathbf{F}}$, we obtain the following result:

5.17 Corollary. *Suppose S is a regular region in $\mathbb{R}^2$ with piecewise smooth boundary ∂S, and let $\mathbf{n}(\mathbf{x})$ be the unit outward normal vector to ∂S at $\mathbf{x} \in \partial S$. Suppose also that $\mathbf{F}$ is a vector field of class C^1 on $\overline{S}$. Then*

$$\int_{\partial S} \mathbf{F} \cdot \mathbf{n}\, ds = \iint_S \left(\frac{\partial F_1}{\partial x_1} + \frac{\partial F_2}{\partial x_2} \right) dA. \tag{5.18}$$

Let us see what Green's theorem says when $\mathbf{F}$ is the gradient of a C^2 function f, so that $F_1 = \partial_1 f$ and $F_2 = \partial_2 f$. Formula (5.13) gives

$$\int_{\partial S} \nabla f \cdot d\mathbf{x} = \iint_S (\partial_1 \partial_2 f - \partial_2 \partial_1 f)\, dA = \iint_S 0\, dA = 0.$$

This is no surprise; it is easy to see directly that the line integral of a gradient over any closed curve vanishes. Indeed, if the curve C is parametrized by $\mathbf{x} = \mathbf{g}(t)$ with $\mathbf{g}(a) = \mathbf{g}(b)$, then by the chain rule,

$$\begin{aligned}\int_C \nabla f \cdot d\mathbf{x} = \int_a^b \nabla f(\mathbf{g}(t)) \cdot \mathbf{g}'(t)\, dt = \int_a^b \frac{d}{dt} f(\mathbf{g}(t))\, dt \\ = f(\mathbf{g}(b)) - f(\mathbf{g}(a)) = 0.\end{aligned}$$

The formula (5.18) gives a more interesting result. $\nabla f \cdot \mathbf{n}$ is the directional derivative of f in the outward normal direction to ∂S, or **normal derivative** of f on ∂S, often denoted by $\partial f / \partial n$; and (29) says that

$$\int_{\partial S} \frac{\partial f}{\partial n}\, ds = \iint_S \left(\frac{\partial^2 f}{\partial x_1^2} + \frac{\partial^2 f}{\partial x_2^2} \right) dA.$$

The integrand on the right is the Laplacian of f, which we encountered in §2.6 and which will play an important role in §5.6.

EXERCISES

1. Evaluate the following line integrals by using Green's theorem.
   a. The integral in Exercise 5c in §5.1.
   b. The integral in Exercise 6c in §5.1.
   c. $\int_C [(x^2 + 10xy + y^2)\, dx + (5x^2 + 5xy) dy]$, where C is the square with vertices $(0,0)$, $(2,0)$, $(0,2)$, and $(2,2)$, oriented counterclockwise.

d. $\int_{\partial S}(3x^2 \sin y^2\, dx + 2x^3 y \cos y^2\, dy)$, where S is any regular region with piecewise smooth boundary.

2. Let S be the annulus $1 \le x^2 + y^2 \le 4$. Compute $\int_{\partial S}(xy^2\, dy - x^2 y\, dx)$, both directly and by using Green's theorem.

3. Find the positively oriented simple closed curve C that maximizes the line integral $\int_C [y^3\, dx + (3x - x^3)\, dy]$.

4. Use Green's theorem as in Example 3 to calculate the area under one arch of the cycloid described parametrically by $x = R(t - \sin t)$, $y = R(1 - \cos t)$.

5. Let $S = \{(x, y) : a \le x \le b,\ 0 \le y \le f(x)\}$, where f is a nonnegative C^1 function on $[a, b]$. Explain how the formula $A = -\int_{\partial S} y\, dx$ for the area of S in Example 3 leads to the familiar formula $A = \int_a^b f(x)\, dx$.

6. Let S be a regular region in $\mathbb{R}^2$ with piecewise smooth boundary, and let f and g be functions of class C^2 on $\overline{S}$. Show that

$$\int_{\partial S} f \frac{\partial g}{\partial n}\, ds = \iint_S \left[f(\partial_x^2 g + \partial_y^2 g) + \nabla f \cdot \nabla g \right] dA.$$

7. The point of this exercise is to show how Green's theorem can be used to deduce a special case of Theorem 4.41. Let U, V be connected open sets in $\mathbb{R}^2$, and let $\mathbf{G} : U \to V$ be a one-to-one transformation of class C^1 whose derivative $D\mathbf{G}(\mathbf{u})$ is invertible for all $\mathbf{u} \in U$. Moreover, let S be a regular region in V with piecewise smooth boundary, let A be its area, and let $T = \mathbf{G}^{-1}(S)$.

a. The Jacobian $\det D\mathbf{G}$ is either everywhere positive or everywhere negative on U; why?

b. Suppose $\det D\mathbf{G}(\mathbf{u}) > 0$ for all $\mathbf{u} \in U$. Write $A = \int_{\partial S} y\, dx$ as in Example 3, make a change of variable to transform this line integral into a line integral over ∂T, and apply Green's theorem to deduce that $A = \iint_T \det D\mathbf{G}\, dA$.

c. By a similar argument, show that if $\det D\mathbf{G}(\mathbf{u}) < 0$ for all $\mathbf{u} \in U$, then $A = -\iint_T \det D\mathbf{G}\, dA = \iint_T |\det D\mathbf{G}|\, dA$. Where does the minus sign come from?

5.3 Surface Area and Surface Integrals

In this section we discuss integrals of functions and vector fields over smooth surfaces in $\mathbb{R}^3$. Like line integrals, surface integrals come in two varieties, unoriented and oriented. On a curve the orientation is a matter of deciding which direction along a curve is "positive"; on a surface it is a matter of deciding which side of the surface is the "positive" side. The convenient way of specifying the orientation of

Finally, as a practical matter we need to extend the ideas in this section from smooth surfaces to piecewise smooth surfaces. Giving a satisfactory general definition of a "piecewise smooth surface" is a rather messy business, and we shall not attempt it. For our present purposes, it will suffice to assume that the surface S under consideration is the union of finitely many pieces $S_1, \ldots, S_k$ that satisfy the following conditions:

i. Each S_j admits a smooth parametrization as discussed above.

ii. The intersections $S_i \cap S_j$ are either empty or finite unions of smooth curves.

We then define integration over S in the obvious way:

$$\iint_S f\,dA = \sum_{j=1}^{k} \iint_{S_j} f\,dA.$$

Condition (ii) guarantees that the parts of S that are counted more than once on the right, namely the intersections $S_i \cap S_j$, contribute nothing to the integral, by Propositions 4.19 and 4.22.

EXAMPLE 3.

a. Let S be the surface of a cube; then we can take $S_1, \ldots, S_6$ to be the faces of the cube.
b. Let S be the surface of the cylindrical solid $\{(x, y, z) : x^2 + y^2 \leq 1,\ |z| \leq 1\}$. We can write $S = S_1 \cup S_2 \cup S_3$ where S_1 and S_2 are the discs forming the top and bottom and S_3 is the circular vertical side. S_1 and S_2 can be parametrized by $(x, y) \to (x, y, 1)$ and $(x, y) \to (x, y, -1)$ with $x^2 + y^2 \leq 1$, and S_3 can be parametrized by $(\theta, z) \to (\cos\theta, \sin\theta, z)$ with $0 \leq \theta < 2\pi$ and $|z| \leq 1$. If one wishes to use only one-to-one parametrizations with compact parameter domains, one can cut S_3 further into two pieces, say the left and right halves defined by $0 \leq \theta \leq \pi$ and $\pi \leq \theta \leq 2\pi$.

Remark. In condition (ii) above, we have in mind that the sets S_j will intersect each other only along their edges, although there is nothing to forbid them from crossing one another. For example, S could be the union of the two spheres $S_1 = \{\mathbf{x} : |\mathbf{x}| = 1\}$ and $S_2 = \{\mathbf{x} : |\mathbf{x} - \mathbf{i}| = 1\}$. This added generality is largely useless but also harmless.

EXERCISES

1. Find the area of the part of the surface $z = xy$ inside the cylinder $x^2 + y^2 = a^2$.

2. Find the area of the part of the surface $z = x^2+y^2$ inside the cylinder $x^2+y^2 = a^2$.

3. Suppose $0 < a < b$. Find the area of the torus obtained by revolving the circle $(x-b)^2 + z^2 = a^2$ in the xz-plane about the z axis. (*Hint:* The torus may be parametrized by $x = (b + a\cos\varphi)\cos\theta$, $y = (b + a\cos\varphi)\sin\theta$, $z = a\sin\varphi$, with $0 \le \varphi, \theta \le 2\pi$.)

4. Find the area of the ellipsoid $(x/a)^2 + (y/a)^2 + (z/b)^2 = 1$.

5. Find the centroid of the upper hemisphere of the unit sphere $x^2 + y^2 + z^2 = 1$.

6. Compute $\iint_S (x^2+y^2)\,dA$ where S is the portion of the sphere $x^2+y^2+z^2 = 4$ with $z \ge 1$.

7. Compute $\iint_S (x^2 + y^2 - 2z^2)\,dA$ where S is the unit sphere. Can you find the answer by symmetry considerations without doing any calculations?

8. Calculate $\iint_S \mathbf{F}\cdot\mathbf{n}\,dA$ for the following $\mathbf{F}$ and S.
   a. $\mathbf{F}(x,y,z) = xz\mathbf{i} - xy\mathbf{k}$; S is the portion of the surface $z = xy$ with $0 \le x \le 1$, $0 \le y \le 2$, oriented so that the normal points upward.
   b. $\mathbf{F}(x,y,z) = x^2\mathbf{i} + z\mathbf{j} - y\mathbf{k}$; S is the unit sphere $x^2+y^2+z^2 = 1$, oriented so that the normal points outward (away from the center).
   c. $\mathbf{F}(x,y,z) = xy\mathbf{i} + z\mathbf{j}$; S is the triangle with vertices $(2,0,0)$, $(0,2,0)$, $(0,0,2)$, oriented so that the normal points upward.
   d. $\mathbf{F}(x,y,z) = z^2\mathbf{k}$; S is the boundary of the region $x^2+y^2 \le 1$, $a \le z \le b$, oriented so that the normal points out of the region. (You should be able to do this in your head.)
   e. $\mathbf{F}(x,y,z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$; S is the boundary of the region $x^2+y^2 \le z \le \sqrt{2 - x^2 - y^2}$, oriented so that the normal points out of the region.

5.4 Vector Derivatives

Let ∇ denote the n-tuple of partial differential operators $\partial_j = \partial/\partial x_j$:

$$\nabla = (\partial_1, \ldots, \partial_n).$$

We are already familiar with this notation in connection with the gradient of a C^1 function on $\mathbb{R}^n$, which is the vector field defined by

$$\operatorname{grad} f = \nabla f = (\partial_1 f, \ldots, \partial_n f).$$

We can also use ∇ to form interesting combinations of the derivatives of a vector field, via the dot and cross product. If $\mathbf{F}$ is a C^1 vector field on an open subset of

It is an important fact that the first two of these always vanish, by the equality of mixed partials:

$$\begin{aligned}(5.30)\quad &\operatorname{curl}(\operatorname{grad} f)\\ &= (\partial_2\partial_3 f - \partial_3\partial_2 f)\mathbf{i} + (\partial_3\partial_1 f - \partial_1\partial_3 f)\mathbf{j} + (\partial_1\partial_2 f - \partial_2\partial_1 f)\mathbf{k} = \mathbf{0}\end{aligned}$$

and

$$\begin{aligned}(5.31)\quad &\operatorname{div}(\operatorname{curl}\mathbf{F})\\ &= \partial_1(\partial_2 F_3 - \partial_3 F_2) + \partial_2(\partial_3 F_1 - \partial_1 F_3) + \partial_3(\partial_1 F_2 - \partial_2 F_1) = 0.\end{aligned}$$

Schematically, we have

$$\begin{matrix}\text{scalar} & \xrightarrow{\text{grad}} & \text{vector} & \xrightarrow{\text{curl}} & \text{vector} & \xrightarrow{\text{div}} & \text{scalar}\\ \text{functions} & & \text{fields} & & \text{fields} & & \text{functions}\end{matrix}$$

and (5.30) and (5.31) say that the composition of two successive mappings is zero.

The third combination, $\operatorname{div}(\operatorname{grad} f)$, which makes sense in any number of dimensions, is of fundamental importance for both physical and purely mathematical reasons. It is called the **Laplacian** of f and is usually denoted by $\nabla^2 f$ or Δf:

$$(5.32)\qquad \nabla^2 f = \Delta f = \operatorname{div}(\operatorname{grad} f) = \partial_1^2 f + \cdots + \partial_n^2 f.$$

The last two combinations are of less interest by themselves, but together they yield the Laplacian for vector fields in $\mathbb{R}^3$:

$$(5.33)\quad \operatorname{grad}(\operatorname{div}\mathbf{F}) - \operatorname{curl}(\operatorname{curl}\mathbf{F}) = \nabla^2\mathbf{F} = (\nabla^2 F_1)\mathbf{i} + (\nabla^2 F_2)\mathbf{j} + (\nabla^2 F_3)\mathbf{k}.$$

The verification of (5.33) is a straightforward but somewhat tedious calculation that we leave to the reader.

EXERCISES

1. Compute the curl and divergence of the following vector fields.
   a. $\mathbf{F}(x,y,z) = xy^2\mathbf{i} + xy\mathbf{j} + xy\mathbf{k}$.
   b. $\mathbf{F}(x,y,z) = (\sin yz)\mathbf{i} + (xz\cos yz)\mathbf{j} + (xy\cos yz)\mathbf{k}$.
   c. $\mathbf{F}(x,y,z) = x^2 z\mathbf{i} + 4xyz\mathbf{j} + (y - 3xz^2)\mathbf{k}$.

2. Compute the Laplacians of the following functions.
   a. $f(x,y) = x^5 - 10x^3y^2 + 5xy^4$.
   b. $f(x,y,z) = xy^2 - 4yz^3$.

c. $f(\mathbf{x}) = |\mathbf{x}|^a$ ($\mathbf{x} \in \mathbb{R}^n \setminus \{\mathbf{0}\}$, $a \in \mathbb{R}$). (*Hint:* Use Exercise 9 in §2.6.)
d. $f(x, y) = \log(x^2 + y^2)$ ($(x, y) \neq (0, 0)$).

3. Let $\mathbf{F}(x, y, z) = x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$. Show that for any $\mathbf{a} \in \mathbb{R}^3$, we have $\operatorname{curl}(\mathbf{a}\times\mathbf{F}) = 2\mathbf{a}$, $\operatorname{div}[(\mathbf{a} \cdot \mathbf{F})\mathbf{a}] = |\mathbf{a}|^2$ and $\operatorname{div}[(\mathbf{a} \times \mathbf{F}) \times \mathbf{a}] = 2|\mathbf{a}|^2$.

4. Prove (5.24) and (5.25).

5. Prove (5.26) and (5.27).

6. Prove (5.28) and (5.29).

7. Prove (5.33).

8. Why is the minus sign in (5.29) there? That is, on grounds of symmetry, without going through any calculations, why must the formula $\operatorname{div}(\mathbf{F} \times \mathbf{G}) = \mathbf{G} \cdot (\operatorname{curl} \mathbf{F}) + \mathbf{F} \cdot (\operatorname{curl} \mathbf{G})$ be wrong?

9. Show that for any C^2 functions f and g, $\operatorname{div}(\operatorname{grad} f \times \operatorname{grad} g) = 0$.

5.5 The Divergence Theorem

The divergence theorem, also known as **Gauss's theorem** or **Ostrogradski's theorem**, is the 3-dimensional analogue of the version (5.18) of Green's theorem; it relates surface integrals over the boundary of a regular region in $\mathbb{R}^3$ to volume integrals over the region itself. The divergence theorem is valid for regions with piecewise smooth boundaries, but we shall allow the meaning of "piecewise smooth" to remain a little vague; see the remarks at the end of §5.3. To formulate precise conditions that encompass all the cases of interest would involve a rather arduous excursion into technicalities, and the more retricted class of regions covered by the following argument suffices for most purposes.

5.34 Theorem (The Divergence Theorem). *Suppose R is a regular region in $\mathbb{R}^3$ with piecewise smooth boundary ∂R, oriented so that the positive normal points out of R. Suppose also that $\mathbf{F}$ is a vector field of class C^1 on R. Then*

$$\iint_{\partial R} \mathbf{F} \cdot \mathbf{n}\, dA = \iiint_R \operatorname{div} \mathbf{F}\, dV. \tag{5.35}$$

Proof. As with Green's theorem, we begin by considering a class of simple regions. We say that R is xy**-simple** if it has the form

$$R = \{(x, y, z) : (x, y) \in W,\ \varphi_1(x, y) \leq z \leq \varphi_2(x, y)\},$$

where W is a regular region in the xy-plane and φ_1 and φ_2 are piecewise smooth functions on W. We define the notions of yz-simple and xz-simple similarly, and we say that R is **simple** if it is xy-simple, yz-simple, and xz-simple.

This approximation becomes better and better as $r \to 0$, and hence

$$\operatorname{div} \mathbf{F}(\mathbf{a}) = \lim_{r \to 0} \frac{3}{4\pi r^3} \iint_{|\mathbf{x}-\mathbf{a}|=r} \mathbf{F} \cdot \mathbf{n}\, dA. \tag{5.36}$$

The integral on the right is the flux of $\mathbf{F}$ across ∂B_r from the inside (B_r) to the outside (the complement of B_r). If we think of the vector field as representing the flow of some substance through space, the integral represents the amount of substance flowing out of B_r minus the amount of substance flowing in; thus, the condition $\operatorname{div} \mathbf{F}(\mathbf{a}) > 0$ means that there is a net outflow near $\mathbf{a}$, in other words, that $\mathbf{F}$ tends to "diverge" from $\mathbf{a}$. (The effect is subtle, though: One has to divide the flux by r^3 in (5.36) to get something that does not vanish in the limit.) In any case, the integral in (5.36) is a geometrically defined quantity that is independent of the choice of coordinates; this gives the promised coordinate-free interpretation of $\operatorname{div} \mathbf{F}$.

Among the important consequences of the divergence theorem are the following identities.

5.37 Corollary (Green's Formulas). *Suppose R is a regular region in $\mathbb{R}^3$ with piecewise smooth boundary, and f and g are functions of class C^2 on $\overline{R}$. Then*

$$\iint_{\partial R} f\nabla g \cdot \mathbf{n}\, dA = \iiint_R (\nabla f \cdot \nabla g + f\nabla^2 g)\, dV, \tag{5.38}$$

$$\iint_{\partial R} (f\nabla g - g\nabla f) \cdot \mathbf{n}\, dA = \iiint_R (f\nabla^2 g - g\nabla^2 f)\, dV. \tag{5.39}$$

Proof. An application of the product rule (5.28) shows that $\operatorname{div}(f\nabla g) = \nabla f \cdot \nabla g + f \cdot \nabla^2 g$, so the divergence theorem applied to $\mathbf{F} = f\nabla g$ yields (5.38). The corresponding equation with f and g switched also holds; by subtracting the latter equation from the former we obtain (5.39). □

The directional derivative $\nabla f \cdot \mathbf{n}$ that occurs in these formulas is called the **outward normal derivative** of f on ∂R and is often denoted by $\partial f/\partial n$.

EXERCISES

In several of these exercises it will be useful to note that if S_r is the sphere of radius r about the origin, the unit outward normal to S_r at a point $\mathbf{x} \in S_r$ is just $r^{-1}\mathbf{x}$. This is geometrically obvious if you think about it a little. Alternatively, since S_r is a level set of the function $|\mathbf{x}|^2 = x^2 + y^2 + z^2$, we know that $\nabla(|\mathbf{x}|^2) = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k} = 2\mathbf{x}$ is normal to S_r, so the unit normal is $|\mathbf{x}|^{-1}\mathbf{x} = r^{-1}|\mathbf{x}|$ for $\mathbf{x} \in S_r$.

1. Use the divergence theorem to evaluate the surface integral $\iint_S \mathbf{F} \cdot \mathbf{n}\, dA$ for the following $\mathbf{F}$ and S, where S is oriented so that the positive normal points out of the region bounded by S.

a. $\mathbf{F}$, S as in Exercise 8b in §5.3.

b. $\mathbf{F}$, S as in Exercise 8e in §5.3.

c. $\mathbf{F}(x, y, z) = x^2\mathbf{i} + y^2\mathbf{j} + z^2\mathbf{k}$; S is the surface of the cube $0 \le x, y, z \le a$.

d. $\mathbf{F}(x, y, z) = (x/a^2)\mathbf{i} + (y/b^2)\mathbf{j} + (z/c^2)\mathbf{k}$; S is the ellipsoid $(x/a)^2 + (y/b)^2 + (z/c)^2 = 1$.

e. $\mathbf{F}(x, y, z) = x^2\mathbf{i} - 2xy\mathbf{j} + z^2\mathbf{k}$; S is the surface of the cylindrical solid $\{(x, y, z) : (x, y) \in W,\ 1 \le z \le 2\}$ where W is a smoothly bounded regular region in the plane with area A.

2. Let $\mathbf{F}(x, y, z) = (x^2 + y^2 + z^2)(x\mathbf{i} + y\mathbf{j} + z\mathbf{k})$ and let S be the sphere of radius a about the origin. Compute $\iint_S \mathbf{F} \cdot \mathbf{n}$ both directly and by the divergence theorem.

3. Let R be a regular region in $\mathbb{R}^3$ with piecewise smooth boundary. Show that the volume of R is $\frac{1}{3} \iint_{\partial R} \mathbf{F} \cdot \mathbf{n}\, dA$ where $\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$.

4. Prove the following integration-by-parts formula for triple integrals:

$$\iiint_R f \frac{\partial g}{\partial x}\, dV = -\iiint_R g \frac{\partial f}{\partial x}\, dV + \iint_{\partial R} f g n_x\, dA,$$

where n_x is the x-component of the unit outward normal to ∂R. (Of course, similar formulas also hold with x replaced by y and z.)

5. Suppose R is a regular region in $\mathbb{R}^3$ with piecewise smooth boundary, and f is a function of class C^2 on $\overline{R}$.

a. Show that $\displaystyle\iint_{\partial R} \frac{\partial f}{\partial n}\, dA = \iiint_R \nabla^2 f\, dV$.

b. Show that if $\nabla^2 f = 0$, then $\displaystyle\iint_{\partial R} f \frac{\partial f}{\partial n}\, dA = \iiint_R |\nabla f|^2\, dV$.

6. Let $\mathbf{x} = (x, y, z)$ and $g(\mathbf{x}) = |\mathbf{x}|^{-1} = (x^2 + y^2 + z^2)^{-1/2}$.

a. Compute $\nabla g(\mathbf{x})$ for $\mathbf{x} \neq \mathbf{0}$.

b. Show that $\nabla^2 g(\mathbf{x}) = 0$ for $\mathbf{x} \neq \mathbf{0}$. (Cf. Exercise 9 in §2.6.)

c. Show by direct calculation that $\iint_S (\partial g/\partial n)\, dA = -4\pi$ if S is any sphere centered at the origin.

d. Since $\partial g/\partial n = \nabla g \cdot \mathbf{n}$ and $\nabla^2 g = \operatorname{div}(\nabla g)$, why do (b) and (c) not contradict the divergence theorem?

e. Show that $\iint_{\partial R} (\partial g/\partial n)\, dA = -4\pi$ if R is any regular region with piecewise smooth boundary whose interior contains the origin. (*Hint:* Consider the region obtained by excising a small ball about the origin from R.)

7. Suppose that f is a C^2 function on $\mathbb{R}^3$ that satisfies Laplace's equation $\nabla^2 f = 0$.

a. By applying (5.39) to f and g, with g as in Exercise 6 and $R = \{\mathbf{x} : \epsilon \leq |\mathbf{x}| \leq r\}$, show that the mean values of f on the spheres $|\mathbf{x}| = r$ and $|\mathbf{x}| = \epsilon$ are equal. (Use Exercises 5a and 6.)

b. Conclude that the mean value of f on any sphere centered at the origin is equal to the value of f at the origin. (*Remark:* There is nothing special about the origin here. By applying this result to $\widetilde{f}(\mathbf{x}) = f(\mathbf{x} + \mathbf{a})$, which also satisfies Laplace's equation, we see that the mean value of f on any sphere is the value of f at the center. The converse is also true; a function that has this mean value property must satisfy Laplace's equation.)

5.6 Some Applications to Physics

In this section we illustrate the uses of the divergence theorem by deriving some important differential equations of mathematical physics. We make a standing assumption that all unspecified mathematical functions that denote physical quantities are smooth enough to ensure the validity of the calculations.

Flow of Material. We have previously alluded to an interpretation of a vector field in terms of material flowing through space. We now develop this idea in more detail.

Suppose there is some substance moving through a region of space — it might be air, water, electric charge, or whatever. The distribution of the substance is given by a *density function* $\rho(\mathbf{x}, t)$; thus $\rho(\mathbf{x}, t)\, dV$ is the amount of substance at time t in a small box of volume dV located at the point $\mathbf{x} = (x, y, z)$. The substance is moving around, so we also have the *velocity field* $\mathbf{v}(\mathbf{x}, t)$ that gives the velocity of the substance at position $\mathbf{x}$ and time t.

Now consider a small bit of oriented surface dS (imagined, not physical) with area dA and normal vector $\mathbf{n}$ located near the point $\mathbf{x}$. (We shall picture dS as a parallelogram, but its exact shape is unimportant.) At what rate does the substance flow through this bit of surface?

First suppose that $\mathbf{n}$ is parallel to the velocity $\mathbf{v} = \mathbf{v}(\mathbf{x}, t)$. We picture a small box with vertical face dS and length $|\mathbf{v}|\, dt$, where dt is a small increment in time, as in Figure 5.8a. We assume that the box is sufficiently small so that that the velocity and density are essentally constant throughout the box during the time interval $(t, t + dt)$. Then the substance that flows through the surface dS in the time interval dt is just the contents of the box at time t. The volume of the box is density ρ, and the current density $\mathbf{J}$. In suitably normalized units, they are

$$\text{(5.50)}\qquad \begin{aligned} \operatorname{div}\mathbf{E} &= 4\pi\rho, \qquad & \operatorname{curl}\mathbf{E} &= -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t}, \\ \operatorname{div}\mathbf{B} &= 0, \qquad & \operatorname{curl}\mathbf{B} &= \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} + \frac{4\pi}{c}\mathbf{J}, \end{aligned}$$

where c is the speed of light. This is not the place for a thorough study of Maxwell's equations and their consequences for physics, but we wish to point out a couple of features of them in connection with the ideas we have been developing. In what follows we shall assume that all functions in question are of class C^2, so that the second derivatives make sense and the mixed partials are equal.

First, Maxwell's equations contain the law of conservation of charge. Indeed, by formula (5.30) we have

$$\frac{\partial \rho}{\partial t} = \frac{1}{4\pi}\operatorname{div}\frac{\partial \mathbf{E}}{\partial t} = \frac{c}{4\pi}\operatorname{div}(\operatorname{curl}\mathbf{B}) - \operatorname{div}\mathbf{J} = -\operatorname{div}\mathbf{J},$$

and this is the conservation law in the form (5.41). Second, in a region of space with no charges or currents ($\rho = 0$ and $\mathbf{J} = \mathbf{0}$), by formula (5.33) we have

$$\nabla^2\mathbf{E} = \nabla(\operatorname{div}\mathbf{E}) - \operatorname{curl}(\operatorname{curl}\mathbf{E}) = \mathbf{0} + \frac{1}{c}\operatorname{curl}\frac{\partial \mathbf{B}}{\partial t} = \frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2}$$

and

$$\nabla^2\mathbf{B} = \nabla(\operatorname{div}\mathbf{B}) - \operatorname{curl}(\operatorname{curl}\mathbf{B}) = \mathbf{0} - \frac{1}{c}\operatorname{curl}\frac{\partial \mathbf{E}}{\partial t} = \frac{1}{c^2}\frac{\partial^2 \mathbf{B}}{\partial t^2}.$$

That is, the components of $\mathbf{E}$ and $\mathbf{B}$ all satisfy the differential equation

$$\text{(5.51)}\qquad \nabla^2 f = \frac{1}{c^2}\frac{\partial^2 f}{\partial t^2}.$$

This is the **wave equation**, another of the fundamental equations of mathematical physics. It describes the propagation of waves in many different situations; here it concerns electromagnetic radiation — light, radio waves, X-rays, and so on.

EXERCISES

Besides distributions of charge or mass in 3-space, one can consider distributions on surfaces or curves (physically: thin plates or wires). The formula for the associated potential or field is similar to (5.43) except that the triple integral is replaced by a surface or line integral, and the density ρ represents charge or mass per unit area or unit length rather than unit volume. In the following exercises, "uniform" means "of constant density."

1. Consider a uniform distribution of mass on the sphere of radius r about the origin. Show that

a. inside the sphere, the potential is constant and the gravitational field vanishes;

b. outside the sphere; the potential and field are the same as if the entire mass were located at the origin.

2. Consider a uniform distribution of mass on the solid ball of radius R about the origin. Show that the gravitational field at a point $\mathbf{x}$ is the same as if the mass closer to the origin than $\mathbf{x}$ were all located at the origin and the mass farther from the origin than $\mathbf{x}$ (if any) were absent. (Use Exercise 1.)

3. Consider a uniform distribution of charge on the z-axis, with density ρ (charge per unit length).

a. Compute the electric field generated by this distribution. (The relevant formula is similar to (5.43), but $1/|\mathbf{p}-\mathbf{x}|$ is replaced by the negative of its gradient with respect to $\mathbf{x}$, namely, $(\mathbf{x}-\mathbf{p})/|\mathbf{x}-\mathbf{p}|^3$.)

b. Show that the modification of (5.43) that presumably gives the potential for this charge distribution is a divergent integral.

c. To resolve the difficulty presented by (b), we make use of the fact that the defining property of the potential u, namely $\nabla u = -\mathbf{E}$, only determines u up to an additive constant, so we may subtract constants from u without affecting the physics. Consider instead a uniform distribution of charge on the interval $[-R, R]$ on the z-axis with density ρ. Compute the potential u_R generated by this distribution, and show that $u_R - 2\rho\log R$ converges as $R \to \infty$ to a function whose gradient is the negative of the field found in (a). (This sort of removal of divergences by "subtracting off infinite constants" is common in quantum field theory, where it is known as *renormalization*.)

4. Prove the following two-dimensional analogue of Theorem 5.46: Suppose ρ is a function of class C^2 on $\mathbb{R}^2$ that vanishes outside a bounded set, and let

$$u(\mathbf{x}) = \int \rho(\mathbf{x}+\mathbf{y}) \log|\mathbf{y}|\, d^2\mathbf{y}.$$

Then u is of class C^2 and $\nabla^2 u = 2\pi\rho$. (The proof is very similar to that of Theorem 5.46; see Exercise 2d in §5.4.)

5.7 Stokes's Theorem

Stokes's theorem is the generalization of Green's theorem in which the plane is replaced by a curved surface. The precise setting is as follows.

Since $\mathbf{u}$ is the normal to D_ϵ, Stokes's theorem gives

$$(\operatorname{curl}\mathbf{F}(\mathbf{a}))\cdot\mathbf{u} = \lim_{\epsilon\to 0}\frac{1}{\pi\epsilon^2}\int_{C_\epsilon}\mathbf{F}\cdot d\mathbf{x}, \tag{5.58}$$

where C_ϵ is the circle of radius ϵ about $\mathbf{a}$ in the plane perpendicular to $\mathbf{u}$, traversed counterclockwise as viewed from the side on which $\mathbf{u}$ lies. This is the promised coordinate-free description of $\operatorname{curl}\mathbf{F}$.

If we think of $\mathbf{F}$ as a force field, $\int_{C_\epsilon}\mathbf{F}\cdot d\mathbf{x}$ is the work done by $\mathbf{F}$ on a particle that moves around C_ϵ. Thus (5.58) says that $(\operatorname{curl}\mathbf{F}(\mathbf{a}))\cdot\mathbf{u}$ represents the tendency of the force $\mathbf{F}$ to push the particle around C_ϵ, counterclockwise if $(\operatorname{curl}\mathbf{F}(\mathbf{a}))\cdot\mathbf{u}$ is positive and clockwise if it is negative (as viewed from the $\mathbf{u}$-side).

EXERCISES

1. Use Stokes's theorem to calculate $\int_C[(x-z)\,dx + (x+y)\,dy + (y+z)\,dz]$ where C is the ellipse where the plane $z=y$ intersects the cylinder $x^2+y^2=1$, oriented counterclockwise as viewed from above.

2. Use Stokes's theorem to evaluate $\int_C[y\,dx + y^2\,dy + (x+2z)\,dz]$ where C is the curve of intersection of the sphere $x^2+y^2+z^2=a^2$ and the plane $y+z=a$, oriented counterclockwise as viewed from above.

3. Given any nonvertical plane P parallel to the x-axis, let C be the curve of intersection of P with the cylinder $x^2+y^2=a^2$. Show that $\int_C[(yz-y)\,dx + (xz+x)\,dy] = 2\pi a^2$.

4. Evaluate $\iint_S \operatorname{curl}\mathbf{F}\cdot\mathbf{n}\,dA$ where $\mathbf{F}(x,y,z) = y\mathbf{i} + (x-2x^3z)\mathbf{j} + xy^3\mathbf{k}$ and S is the upper half of the sphere $x^2+y^2+z^2=a^2$.

5. Let $\mathbf{F}(x,y,z) = 2x\mathbf{i} + 2y\mathbf{j} + (x^2+y^2+z^2)\mathbf{k}$ and let S be the lower half of the ellipsoid $(x^2/4)+(y^2/9)+(z^2/27)=1$. Use Stokes's theorem to calculate the flux of $\operatorname{curl}\mathbf{F}$ across S from the lower side to the upper side.

6. Define the vector field $\mathbf{F}$ on the complement of the z-axis by $\mathbf{F}(x,y,z) = (-y\mathbf{i}+x\mathbf{j})/(x^2+y^2)$.

a. Show that $\operatorname{curl}\mathbf{F} = \mathbf{0}$.

b. Show by direct calculation $\int_C\mathbf{F}\cdot d\mathbf{x} = 2\pi$ for any horizontal circle C centered at a point on the z-axis.

c. Why do (a) and (b) not contradict Stokes's theorem?

7. Let C_r denote the circle of radius r about the origin in the xz-plane, oriented counterclockwise as viewed from the positive y-axis. Suppose $\mathbf{F}$ is a C^1 vector field on the complement of the y-axis in $\mathbb{R}^3$ such that $\int_{C_1}\mathbf{F}\cdot d\mathbf{x} = 5$ and $\operatorname{curl}\mathbf{F}(x,y,z) = 3\mathbf{j} + (z\mathbf{i}-x\mathbf{k})/(x^2+z^2)^2$. Compute $\int_{C_r}\mathbf{F}\cdot d\mathbf{x}$ for every r.

8. Let S be a smooth oriented surface in $\mathbb{R}^3$ with piecewise smooth, compatibly oriented boundary ∂S. Suppose f is C^1 and g is C^2 on some open set containing S. Show that

$$\int_{\partial S} f\nabla g \cdot d\mathbf{x} = \iint_S (\nabla f \times \nabla g) \cdot \mathbf{n}\, dA.$$

5.8 Integrating Vector Derivatives

In this section we study the question of solving the equations

$$\operatorname{grad} f = \mathbf{G}, \qquad \operatorname{curl}\mathbf{F} = \mathbf{G}, \qquad \operatorname{div}\mathbf{F} = g$$

for f or $\mathbf{F}$, given g or $\mathbf{G}$. We first consider the equation $\nabla f = \mathbf{G}$, and we begin with a simple and useful result:

5.59. Proposition. *Suppose* $\mathbf{G}$ *is a continuous vector field on an open set* R *in* $\mathbb{R}^n$. *The following two conditions are equivalent:*

a. If C_1 *and* C_2 *are any two oriented curves in* R *with the same initial point and the same final point, then* $\int_{C_1} \mathbf{G} \cdot d\mathbf{x} = \int_{C_2} \mathbf{G} \cdot d\mathbf{x}$.

b. If C *is any closed curve in* R, $\int_C \mathbf{G} \cdot d\mathbf{x} = 0$.

Proof. (a) implies (b): Suppose C starts and ends at $\mathbf{a}$. Then C has the same initial and final point as the "constant curve" C_2 described by $\mathbf{x}(t) \equiv \mathbf{a}$, and obviously $\int_{C_2} \mathbf{G} \cdot d\mathbf{x} = 0$ since $d\mathbf{x} = \mathbf{0}$ on C_2.

(b) implies (a): Suppose C_1 and C_2 start at $\mathbf{a}$ and end at $\mathbf{b}$. Let C be the closed curve obtained by following C_1 from $\mathbf{a}$ to $\mathbf{b}$ and then C_2 backwards from $\mathbf{b}$ to $\mathbf{a}$. Then $0 = \int_C \mathbf{G} \cdot d\mathbf{x} = \int_{C_1} \mathbf{G} \cdot d\mathbf{x} - \int_{C_2} \mathbf{G} \cdot d\mathbf{x}$. □

A vector field $\mathbf{G}$ that satisfies (a) and (b) is called **conservative** in the region R. (The word "conservative" has to do with conservation of energy. If we interpret $\mathbf{G}$ as a force field, condition (b) says that the force does no net work on a particle that returns to its starting point.) A good deal of mathematical physics is based on the following characterization of conservative vector fields:

5.60. Proposition. *A continuous vector field* $\mathbf{G}$ *in an open set* $R \subset \mathbb{R}^n$ *is conservative if and only if* $\mathbf{G}$ *is the gradient of a* C^1 *function* f *on* R.

Proof. If $\mathbf{G} = \nabla f$ and C is a closed curve parametrized by $\mathbf{x} = \mathbf{g}(t)$, $a \le t \le b$, by the chain rule we have

$$\int_C \nabla f \cdot d\mathbf{x} = \int_a^b \nabla f(\mathbf{g}(t)) \cdot \mathbf{g}'(t)\, dt = \int_a^b \frac{d}{dt} f(\mathbf{g}(t))\, dt$$
$$= f(\mathbf{g}(b)) - f(\mathbf{g}(a)) = 0$$

field $\mathbf{E}$ vanishes only when there are no time-varying magnetic fields present. Only in this case is $\mathbf{E}$ the gradient of a potential function. However, $\operatorname{div} \mathbf{B} = 0$ always (this expresses the fact that there are no "magnetic charges"), so $\mathbf{B}$ is the curl of a vector potential $\mathbf{A}$. We then have

$$\operatorname{curl}\left(\mathbf{E} + \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t}\right) = \operatorname{curl} \mathbf{E} + \frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} = \mathbf{0},$$

so $\mathbf{E} + c^{-1}\partial_t \mathbf{A}$ *is* the gradient of a function $-\varphi$. The four-component quantity $(\varphi, \mathbf{A}) = (\varphi, A_1, A_2, A_3)$ is called the *electromagnetic 4-potential.* It is best regarded as a vector in 4-dimensional space-time, with φ being the time component, in the context of special relativity.

EXERCISES

1. Determine whether each of the following vector fields is the gradient of a function f, and if so, find f. The vector fields in (a)–(c) are on $\mathbb{R}^2$; those in (d)–(f) are on $\mathbb{R}^3$, and the one in (g) is on $\mathbb{R}^4$. In all cases $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$, and $\mathbf{l}$ denote unit vectors along the positive x-, y-, z-, and w-axes.

a. $\mathbf{G}(x, y) = (2xy + x^2)\mathbf{i} + (x^2 - y^2)\mathbf{j}$.

b. $\mathbf{G}(x, y) = (3y^2 + 5x^4 y)\mathbf{i} + (x^5 - 6xy)\mathbf{j}$.

c. $\mathbf{G}(x, y) = (2e^{2x}\sin y - 3y + 5)\mathbf{i} + (e^{2x}\cos y - 3x)\mathbf{j}$

d. $\mathbf{G}(x, y, z) = (yz - y\sin xy)\mathbf{i} + (xz - x\sin xy + z\cos yz)\mathbf{j} + (xy + y\cos yz)\mathbf{k}$.

e. $\mathbf{G}(x, y, z) = (y - z)\mathbf{i} + (x - z)\mathbf{j} + (x - y)\mathbf{k}$

f. $\mathbf{G}(x, y, z) = 2xy\mathbf{i} + (x^2 + \log z)\mathbf{j} + ((y + 2)/z)\mathbf{k} \quad (z > 0)$.

g. $\mathbf{G}(x, y, z, w) = (xw^2 + yzw)\mathbf{i} + (xzw + yz^2 - 2e^{2y+z})\mathbf{j} + (xyw + y^2 z - e^{2y+z} - w\sin zw)\mathbf{k} + (xyz + x^2 w - z\sin zw)\mathbf{l}$.

2. Determine whether each of the following vector fields is the curl of a vector field $\mathbf{F}$, and if so, find such an $\mathbf{F}$.

a. $\mathbf{G}(x, y, z) = (x^3 + yz)\mathbf{i} + (y - 3x^2 y)\mathbf{j} + 4y^2\mathbf{k}$.

b. $\mathbf{G}(x, y, z) = (xy + z)\mathbf{i} + xz\mathbf{j} - (yz + x)\mathbf{k}$.

c. $\mathbf{G}(x, y, z) = (xe^{-x^2 z^2} - 6x)\mathbf{i} + (5y + 2z)\mathbf{j} + (z - ze^{-x^2 z^2})\mathbf{k}$.

3. Let R be a bounded convex open set in $\mathbb{R}^3$. Show that for any C^2 vector field $\mathbf{H}$ on $\overline{R}$ there exist a C^2 function f and a C^2 vector field $\mathbf{G}$ such that $\mathbf{H} = \operatorname{grad} f + \operatorname{curl} \mathbf{G}$. (*Hint:* Solve $\nabla^2 f = \operatorname{div} \mathbf{H}$.)

4. Let $\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j}$ be a C^1 vector field on $S = \mathbb{R}^2 \setminus \{(0, 0)\}$ such that $\partial_1 F_2 = \partial_2 F_1$ on S (but $\mathbf{F}$ may be singular at the origin).

a. Let C_r be the circle of radius r about the origin, oriented counterclockwise. Show that $\int_{C_r} \mathbf{F} \cdot d\mathbf{x}$ is a constant α that does not depend on r. (*Hint:* Consider the region between two circles.)
b. Show that $\int_C \mathbf{F} \cdot d\mathbf{x} = \alpha$ for *any* simple closed curve C, oriented counterclockwise, that encircles the origin.
c. Let $\mathbf{F}_0 = (x\mathbf{j} - y\mathbf{i})/(x^2 + y^2)$ as in Example 1. Show that $\mathbf{F} - (\alpha/2\pi)\mathbf{F}_0$ is the gradient of a function on S. (Thus, all curl-free vector fields on S that are not gradients can be obtained from $\mathbf{F}_0$ by adding gradients.)

5.9 Higher Dimensions and Differential Forms

Green's theorem has to do with integrals of vector fields in the plane, and the divergence theorem and Stokes's theorem have do do with integrals of vector fields in 3-space. What happens in dimension n? There are a couple of things we can say without too much additional explanation.

First, the obvious analogue of the divergence theorem holds in $\mathbb{R}^n$ for any $n > 1$. To wit, if R is a regular region in $\mathbb{R}^n$ bounded by a piecewise smooth hypersurface ∂R, and $\mathbf{F}$ is a C^1 vector field on R, then

$$\int \cdots \int_{\partial R} \mathbf{F} \cdot \mathbf{n}\, dV^{n-1} = \iint \cdots \int_R \operatorname{div} \mathbf{F}\, dV^n.$$

Here dV^n is the n-dimensional volume element in $\mathbb{R}^n$ and dV^{n-1} is the $(n-1)$-dimensional "area" element on ∂R. The "vector area element" $\mathbf{n}\, dV^{n-1}$ is given by a formula analogous to the one in $\mathbb{R}^3$. Namely, if (part of) ∂R is parametrized by $\mathbf{x} = \mathbf{G}(u_1, \ldots, u_{n-1})$, then

$$\mathbf{n}\, dV^{n-1} = \det \begin{pmatrix} \mathbf{e}_1 & \cdots & \mathbf{e}_n \\ \partial_1 G_1 & \cdots & \partial_1 G_n \\ \vdots & & \vdots \\ \partial_1 G_{n-1} & \cdots & \partial_{n-1} G_n \end{pmatrix} du_1 \cdots du_{n-1},$$

where $\mathbf{e}_1, \ldots, \mathbf{e}_n$ are the standard basis vectors for $\mathbb{R}^n$. (The reader may verify that in the case $n = 2$, these formulas yield Green's theorem in the form (5.18).)

Second, the analogue of the divergence theorem in dimension 1 is just the fundamental theorem of calculus:

$$f(b) - f(a) = \int_{[a,b]} f'(t)\, dt.$$

On the real line, vector fields are the same thing as functions, and the divergence of a vector field is just the derivative of a function. A regular region in $\mathbb{R}$ is an interval